*On the Definition of Reducible Hypercomplex Number Systems.**

By Saul Epsteen and Heman Burr Leonard.

§1.—*Preliminary Remarks.*

A hypercomplex number system is said to be reducible† when, by a proper choice of the units

$$E \equiv E_j\ E_k \equiv e_1 \ \ldots\ldots\ e_m e_{m+1} \ \ldots\ldots\ e_n, \text{ with the relations } e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3},$$

the following conditions are fulfilled:

C_1), E_j forms a system by itself,

$$e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3}, \qquad (\gamma_{j_1 j_2 k} = 0);$$

C_2), E_k forms a system by itself,

$$e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3}, \qquad (\gamma_{k_1 k_2 j} = 0);$$

C_{jk}), $\qquad e_j e_k = 0, \quad j = 1, \ldots\ldots, m, \qquad (\gamma_{jki} = 0);$

C_{kj}), $\qquad e_k e_j = 0, \quad k = m+1, \ldots\ldots, n, \quad (\gamma_{kji} = 0).$

The system E is supposed to be associative. We add moreover conditions concerning division.

A), associativity;

C_r), right-hand division possible and unique, that is, not every X is a right-hand divisor of zero;

C_l), left-hand division possible and unique, that is, not every X is a left-hand divisor of zero.

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* Read before the Chicago Section of the American Mathematical Society, April 2, 1904.

† Benj. Peirce, American Journal of Mathematics, vol. 4 (1881), p. 100; Scheffers, Mathematische Annalen, vol. 39 (1891), p. 317.

In the previous paper of the same title* it was shown that the conditions C_1 and C_2 are consequences of

(1) A, C_{jk}, C_{kj}, C_r, or of
(2) A, C_{jk}, C_{kj}, C_l.

In (1) and (2) *the assumptions* A, C_r, C_l *are characterized by the fact that they refer to the system* E *as a whole.*

Adopting suggestions of Professor E. H. Moore the associativity condition A is separated into eight parts and the conditions C_{jk} and C_{kj} are each separated into two parts.

The general number $X_p = \sum_{i_1=1}^{n} x_{pi_1}e_{i_1}$ of the system E is the sum of two components

$$X_p = J_p + K_p = \sum_{j_1=1}^{m} x_{pj_1}e_{j_1} + \sum_{k_1=m+1}^{n} x_{pk_1}e_{k_1}.$$

The condition A: $(X_1X_2)X_3 = X_1(X_2X_3)$ is equivalent to the following eight:

$$\begin{aligned}
A_1), &\quad (J_1 J_2) J_3 = J_1 (J_2 J_3);\\
A_2), &\quad (K_1 K_2) K_3 = K_1 (K_2 K_3);\\
A_3), &\quad (K_1 J_1) J_2 = K_1 (J_1 J_2);\\
A_4), &\quad (J_1 K_1) K_2 = J_1 (K_1 K_2);\\
A_5), &\quad (J_1 K_1) J_2 = J_1 (K_1 J_2);\\
A_6), &\quad (K_1 J_1) K_2 = K_1 (J_1 K_2);\\
A_7), &\quad (J_1 J_2) K_1 = J_1 (J_2 K_1);\\
A_8), &\quad (K_1 K_2) J_1 = K_1 (K_2 J_1).
\end{aligned}$$

In general $J_1K_1 = J_2 + K_2$. The condition C_{jk} says that $\left\{\begin{matrix} J_2 = 0 \\ K_2 = 0 \end{matrix}\right\}$ simultaneously. Thus it is seen that C_{jk} is equivalent to the two conditions:

$$\begin{aligned}
C^j_{jk}), &\quad J_1K_1 = 0 + K_2 = K_2;\\
C^k_{jk}), &\quad J_1K_1 = J_2 + 0 = J_2.
\end{aligned}$$

In general $K_3J_3 = J_4 + K_4$. Similarly then, C_{kj} says that $\left\{\begin{matrix} J_4 = 0 \\ K_4 = 0 \end{matrix}\right\}$

* Epsteen, Transactions of the American Mathematical Society, vol. 5 (1904), p. 105.

simultaneously; and thus C_{kj} is equivalent to the two conditions:

$$C^j_{kj}),\qquad K_3J_3 = 0 + K_4 = K_4\,;$$
$$C^k_{kj}),\qquad K_3J_3 = J_4 + 0 = J_4\,.$$

The condition for right-hand division may be expressed thus:

C_r), not every X is a right-hand divisor of zero; hence, an X exists such that

$$X_1X = 0,\ \text{only if}\ X_1 = 0\,.$$

The condition for left-hand division may be expressed thus:

C_l), not every X is a left-hand divisor of zero; hence, an X exists such that

$$XX_1 = 0\,,\ \text{only if}\ X_1 = 0.$$

When the condition C_1 is not true, then there are at least two numbers of the set of units E_j, namely J_1 and J_2, such that their product

$$J_1J_2 = J_3 + K_3 \qquad\qquad (K_3 \neq 0).$$

And likewise if C_2 is not true, there are at least two numbers K_1 and K_2 of the set of units E_k, such that their product

$$K_1K_2 = J_3 + K_3 \qquad\qquad (J_3 \neq 0).$$

The condition C^j_r which is employed in this paper means that

C^j_r), not every J is a right-hand divisor of zero in the set E_j; hence, a J exists such that

$$J_1J = 0,\ \text{only if}\ J_1 = 0\,.$$

Similarly C^j_l means that

C^j_l), not every J is a left-hand divisor of zero in the set E_j; hence, a J exists such that

$$JJ_1 = 0,\ \text{only if}\ J_1 = 0\,.$$

From the preceding the meaning of C^k_r and C^k_l is evident.

We have, therefore, to consider in this paper the following twenty conditions:

A_1), $(J_1 J_2) J_3 = J_1 (J_2 J_3)$;
A_2), $(K_1 K_2) K_3 = K_1 (K_2 K_3)$;
A_3), $(K_1 J_1) J_2 = K_1 (J_1 J_2)$;
A_4), $(J_1 K_1) K_2 = J_1 (K_1 K_2)$;
A_5), $(J_1 K_1) J_2 = J_1 (K_1 J_2)$;
A_6), $(K_1 J_1) K_2 = K_1 (J_1 K_2)$;
A_7), $(J_1 J_2) K_1 = J_1 (J_2 K_1)$;
A_8), $(K_1 K_2) J_1 = K_1 (K_2 J_1)$;

C_1), E_j is closed under multiplication, that is $J_1 J_2 = J_3$;
C_2), E_k is closed under multiplication, that is $K_1 K_2 = K_3$;

C^j_{jk}), $J_1 K_1 = K_2 \quad (J_2 = 0)$;
C^k_{jk}), $J_1 K_1 = J_2 \quad (K_2 = 0)$;
C^j_{kj}), $K_1 J_1 = K_2 \quad (J_2 = 0)$;
C^k_{kj}), $K_1 J_1 = J_2 \quad (K_2 = 0)$;

C_r), not every X is a right-hand divisor of zero;
C_l), not every X is a left-hand divisor of zero;
C^j_r), not every J is a right-hand divisor of zero in E_j;
C^j_l), not every J is a left-hand divisor of zero in E_j;
C^k_r), not every K is a right-hand divisor of zero in E_k;
C^k_l), not every K is a left-hand divisor of zero in E_k.

§2.—*Dependencies.*

We prove the following dependencies:

D_1). The condition C_1 says that $J_1 J_2 = J_3$. Suppose that

$$J_1 J_2 = J_3 + K_3. \tag{3}$$

Multiplying both sides on the right by K

$$(J_1 J_2) K = J_3 K + K_3 K.$$

By A_7, C^j_{jk}, C^k_{jk} this becomes

$$0 = K_3 K. \tag{4}$$

By C^j_{kj}, C^k_{kj} $\quad 0 = K_3 J.$

Adding we obtain $\quad 0 = K_3 (J + K) = K_3 X.$

Hence by C_r $\quad K_3 = 0.$

The product $J_1 J_2$ being equal to J_3 proves the condition C_1.

TABLE I.

Notation.	Assumptions.	Consequence.
D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r	C_1
D_2	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l	C_1
D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r	C_1
D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l	C_1
D_5	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r	C_2
D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l	C_2
D_7	A_8, C^j_{kj}, C^k_{kj} C^j_r	C_2
D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l	C_2
D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}
D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}
D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}
D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}

D_2). Instead of multiplying (3) *on the right* by K and adding $K_1J=0$, we multiply it *on the left* by K and add $JK_1=0$. In this way it is easily seen that C_1 is a consequence of A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, and C_l.

D_3). From equation (4) and C^k_r it follows that $K_1=0$. Hence C_1 is a consequence of A_7, C^j_{jk}, C^k_{jk}, C^k_r.

D_4). From the equation $KK_1=0$, which arises in the proof of D_2, it follows from C^k_l that $K_1=0$. Hence C_1 is a consequence of A_3, C^j_{kj}, C^k_{kj}, C^k_l.

D_5). This follows by interchanging j and k in D_1.

D_6). This follows by interchanging j and k in D_2.

D_7). This follows by interchanging j and k in D_3.

D_8). This follows by interchanging j and k in D_4.

D_9). The condition C^j_{jk} says that $J_1K_1 = K_2$. Suppose that

$$J_1K_1 = J_2 + K_2. \tag{5}$$

Multiplying both sides on the right by J

$$(J_1K_1)\,J = J_2J + K_2J.$$

By A_5, C^j_{kj}, C^k_{kj} this becomes

$$0 = J_2J. \tag{6}$$

By C^j_r we see that $J_2 = 0$. Therefore C^j_{jk} is a consequence of A_5, C^j_{kj}, C^k_{kj}, and C^j_r.

D_{10}). Instead of multiplying (5) *on the right* by J, we multiply it *on the left* by K. In this way it is readily seen that C^k_{jk} is a consequence of A_6, C^j_{kj}, C^k_{kj}, and C^k_l.

D_{11}). This follows by interchanging j and k in D_9.

D_{12}). This follows by interchanging j and k in D_{10}.

The conditions C_1, C_2, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, cannot be derived from any set of conditions other than the above.

§3.—*Definitions of Reducibility by Independent Assumptions.*

From Table I. it can be seen that there are seventy-eight different ways of defining the reducibility of a hypercomplex number system.

I.—The following eight definitions of reducibility are the only ones in which the division assumptions are on the set E as a whole.*

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* All the definitions of this paper contain one or more division assumptions. There is only one definition, the classical one of Peirce-Scheffers, that does not contain a division assumption.

TABLE II.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3)		(6) Proved by (3) and (4).
			(4)	(5)	
R_1	D_6	$A_4,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_l$		C_2	
	D_2	$A_3,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_l$		C_1	
R_2	D_6	$A_4,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_l$		C_2	
	D_1	$A_7,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_r$		C_1	
R_3	D_6	$A_4,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_l$		C_2	
	....	C_1			
R_4	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_r$		C_2	
	D_2	$A_3,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_l$		C_1	
R_5	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_r$		C_2	
	D_1	$A_7,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_r$		C_1	
R_6	D_5	$A_8,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_r$		C_2	
	....	C_1			
R_7	D_2	$A_3,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_l$		C_1	
	....	C_2			
R_8	D_1	$A_7,\ C^j_{jk},\ C^k_{jk},\ C^j_{kj},\ C^k_{kj},\ C_r$		C_1	
	....	C_2			

II.—The following thirty-eight definitions of reducibility are definitions in which the division assumptions are solely on the subsets E_j, E_k.

TABLE III.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
R_9	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_4	A_3, C^k_l	" "		C_1
R_{10}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
R_{11}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C_1			
R_{12}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, C^j_r	" "		C_2
	D_4	A_3, C^k_l	" "		C_1
R_{13}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, C^j_r	" "		C_2
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3): (4)	Proved by (3): (5)	(6) Proved by (3) and (4).
R_{14}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C_2			
R_{15}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	....	C_1			
	....	C_2			
R_{16}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^k_{kj}			
R_{17}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C^k_{kj}			
	....	C_1			
R_{18}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^k_{kj}			
	....	C_2			
R_{19}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_1			
	....	C_2			
R_{20}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{21}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	....	C^j_{kj}			
	....	C_1			
R_{22}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			
	....	C_2			
R_{23}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	....	C^j_{kj}			
	....	C_1			
	....	C_2			
R_{24}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_8	A_4, C^j_l	" "		C_2
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{25}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_8	A_4, C^j_l	" "		C_2
	D_3	A_7, C^k_r	" "		C_1
R_{26}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{27}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}	C_2	
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r			
	D_3	A_7, C^k_r	" "		C_1
R_{28}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}	C_2	
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r			
	....	C_1			
R_{29}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C_2			
R_{30}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	....	C_1			
	....	C_2			
R_{31}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
R_{32}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C^j_{jk}			
	....	C_1			

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
R_{33}	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
	....	C_2			
	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	....	C^j_{jk}			
R_{34}	....	C_1			
	....	C_2			
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
R_{35}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^k_{jk}			
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
R_{36}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C^k_{jk}			
	....	C_1			
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
R_{37}	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^k_{jk}			
	....	C_2			
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	....	C^k_{jk}			
R_{38}	....	C_1			
	....	C_2			
R_{39}	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	

TABLE III.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
R_{40}	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			
	....	C^k_{kj}			
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
R_{41}	....	C^j_{kj}			
	....	C^k_{kj}			
	....	C_1			
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
R_{42}	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
	....	C^k_{jk}			
R_{43}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	....	C^j_{jk}			
R_{44}	....	C^k_{jk}			
	....	C_1			
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
	....	C^j_{jk}			
R_{45}	....	C^k_{jk}			
	....	C_2			
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
	....	C^j_{kj}			
R_{46}	....	C^k_{kj}			
	....	C_2			

III.—The following thirty-two different ways of defining reducibility are based on assumptions concerning division in the set E and its subsets, E_j, E_k, simultaneously.

TABLE IV.*.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{12}	A_5, C^j_{jk}, C^{jk}_{k}, C^j_l	C^j_{kj}		
R_{47}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_2	A_3, C^j_{jk}, C^k_{jk}, C_l	" "		C_1
	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
R_{48}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_8	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C_r	" "		C_1
	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
R_{49}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, C^j_r	" "		C_2
	D_2	A_3, C^j_{jk}, C^k_{jk}, C_l	" "		C_1
	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
R_{50}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_7	A_8, $*C^j_r$	" "		C_2
	D_1	A_7, C^j_{jk}, C^k_{jk}, C_r	" "		C_1
	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
R_{51}	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C_l	" "		C_2
	D_4	A_3, $*C^k_l$	" "		C_1

* A condition marked with a star is dependent on the other assumptions in the particular definition in which it appears. Thus $*C^j_r$ is dependent upon the other assumptions of R_{50} (§§4–5).

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{52}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C_l	" "		C_2
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	
R_{53}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C_l	" "		C_2
	D_2	A_3, C^j_{jk}, C^k_{jk}, C_l	" "		C_1
R_{54}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_6	A_4, C^j_{jk}, C^k_{jk}, C_l	" "		C_2
	D_1	A_7, C^j_{jk}, C^k_{jk}, C_r	" "		C_1
R_{55}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_5	A_8, C^j_{jk}, C^k_{jk}, C_r	" "		C_2
	D_4	A_3, C^k_l	" "		C_1
R_{56}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_5	A_8, C^j_{jk}, C^k_{jk}, C_r	" "		C_2
	D_3	A_7, C^j_{jk}, C^k_{jk}, C^k_r		C_1	

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3): (4)	Proved by (3): (5)	(6) Proved by (3) and (4).
R_{57}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_5	A_8, C^j_{jk}, C^k_{jk}, C_r	" "		C_2
	D_2	A_3, C^j_{jk}, C^k_{jk}, C_l	" "		C_1
R_{58}	D_{12}	A_5, C^j_{jk}, C^k_{jk}, C^j_l	C^j_{kj}		
	D_{11}	A_6, C^j_{jk}, C^k_{jk}, C^k_r	C^k_{kj}		
	D_5	A_8, C^j_{jk}, C^k_{jk}, C_r	" "		C_2
	D_1	A_7, C^j_{jk}, C^k_{jk}, C_r	" "		C_1
R_{59}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_8	A_4, *C^j_l	" "		C_2
	D_2	A_3, C^j_{kj}, C^k_{kj}, C_l	" "		C_1
R_{60}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_8	A_4, C^j_l	" "		C_2
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1
R_{61}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_2	A_3, C^j_{kj}, C^k_{kj}, C_l	" "		C_1

TABLE IV.—Continued.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{62}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1
R_{63}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{64}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_3	A_7, C^k_r	" "		C_1
R_{65}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_2	A_3, C^j_{kj}, C^k_{kj}, C_l	" "		C_1
R_{66}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_6	A_4, C^j_{kj}, C^k_{kj}, C_l	" "		C_2
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
R_{67}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{68}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_3	A_7, $*C^k_r$	" "		C_1
R_{69}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk}		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_2	A_3, C^j_{kj}, C^k_{kj}, C_l	" "		C_1
R_{70}	D_{10}	A_6, C^j_{kj}, C^k_{kj}, C^k_l	C^k_{jk}		
	D_9	A_5, C^j_{kj}, C^k_{kj}, C^j_r	C^j_{jk},		
	D_5	A_8, C^j_{kj}, C^k_{kj}, C_r	" "		C_2
	D_1	A_7, C^j_{kj}, C^k_{kj}, C_r	" "		C_1
R_{71}	D_8	A_4, C^j_{jk}, C^k_{jk}, $*C^j_l$		C_2	
	D_2	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
R_{72}	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_l		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	

TABLE IV.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3)		(6) Proved by (3) and (4).
			(4)	(5)	
R_{73}	D_7	A_8, C^j_{kj}, C^k_{kj}, C^j_r		C_2	
	D_2	A_3, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_1	
R_{74}	D_7	A_8, C^j_{kj}, C^k_{kj}, $*C^j_r$		C_2	
	D_1	A_7, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_1	
R_{75}	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, $*C^k_l$		C_1	
R_{76}	D_6	A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l		C_2	
	D_3	A_7, C_{jk}, C^k_{jk}, C^k_r		C_1	
R_{77}	D_5	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	D_4	A_3, C^j_{kj}, C^k_{kj}, C^k_l		C_1	
R_{78}	D_5	A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r		C_2	
	D_3	A_7, C^j_{jk}, C^k_{jk}, $*C^k_r$		C_1	

§4.—*Independence Proofs.*

The sets of conditions $R_1, R_2, \ldots, R_{78}$ yield reducibility. It can be seen, however, that there are not enough conditions in any one of these sets to prove the associativity A of the system E. In each case it is necessary to add the conditions A_1, A_2. The set R_1 with the conditions A_1, A_2, adjoined, we designate by R_1^A to indicate that the latter conditions yield both reducibility and associativity. Similarly $R_2^A \equiv [R_2, A_1, A_2]; \ldots; R_{78}^A \equiv [R_{78}, A_1, A_2]$.

Table V. contains proofs of the independence of the conditions in each of the sets $R_1, \ldots, R_{78}$, with the exception of eight—$R_{50}, R_{51}, R_{59}, R_{68}, R_{71}, R_{74}, R_{75}, R_{78}$—in which the subset division assumptions will be shown later to be redundant.

To prove these independencies we employ the following multiplication tables, for which the borders have been omitted:

I.

$$\begin{array}{ccc} e_2 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2;\ E_k = e_3.$

II.

$$\begin{array}{cc} e_1 + e_2 & 0 \\ 0 & e_2 \end{array}$$

$E_j = e_1;\ E_k = e_2.$

III.

$$\begin{array}{ccc} e_1 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2;\ E_k = e_3.$

IV.

$$\begin{array}{ccc} e_1 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1;\ E_k = e_2, e_3.$

V.

$$\begin{array}{cc} e_1 & e_1 \\ e_1 & e_2 \end{array}$$

$E_j = e_1;\ E_k = e_2.$

VI.

$$\begin{array}{ccc} 0 & 0 & e_1 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2;\ E_k = e_3.$

VII.

$$\begin{array}{ccc} e_1 & 0 & e_3 \\ 0 & e_2 & 0 \\ 0 & e_3 & 0 \end{array}$$

$E_j = e_1;\ E_k = e_2, e_3.$

VIII.

$$\begin{array}{ccc} e_1 & 0 & 0 \\ e_2 & 0 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2;\ E_k = e_3.$

IX.

$$\begin{array}{ccc} 0 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2;\ E_k = e_3.$

TABLE V.—(*Independence Proofs*).

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	C_1	C_2	C^j_{jk}	C^k_{jk}	C^j_{kj}	C^k_{kj}	C_l	C_r	C^j_l	C^k_l	C^j_r	C^k_r	Proof.
1	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	I.
2	★	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (1).
3	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	★	II.
4	★	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (3).
5	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	★	III.
6	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	★	IV.
7	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	Interchange j, k in (6).
8	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	Interchange j, k in (5).
9	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	★	V.
10	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	VI.
11	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	Interchange j, k in (9).
12	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	VII.
13	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	★	★	★	Interchange j, k in (12).
14	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	Interchange j, k in (10).
15	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	VIII.
16	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	★	Interchange j, k in (15).
17	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	IX.
18	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	Interchange j, k in (17).

According to $R_1{}^A$ the system E is reducible and associative under the assumptions A_1, A_2, A_3, A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l.

That A_1 is independent of the other conditions in $R_1{}^A$ is shown in row 1 of Table V.

The independence of A_2 is shown in row 2.

Similarly the independence of any one of the conditions of the others can be seen by looking down its column in Table V. for the square marked i. In some cases, as in column A_5, the letter i occurs in both row 5 and row 6. In order to prove A_5 independent of the other conditions in $R_9{}^A$ we use row 6; but

to prove A_5 independent of the other conditions in R^4_{24}, row 5 must be used. Likewise there can be no confusion in similar cases—such for example as in row 7 and row 8.

§5.—*Dependence proofs* (R_{50}, R_{51}, R_{59}, R_{68}, R_{71}, R_{74}, R_{75}, R_{78}).

1. In R_{50}* the assumptions are not independent. We proceed to prove that C^j_r *is a consequence of the others,* which are mutually independent.

We know by Table I., D_{12}, D_{11}, that C^j_{kj} and C^k_{kj} are consequences of the assumptions of R_{50} (C^j_r being omitted). Table I. D_1, D_5 show that C_1 and C_2 are consequences of A_7, A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r, and, therefore, can be derived from the above-mentioned independent conditions of R_{50}. According to the condition C_r, there exist a J and a K such that

$$(J_1 + K_1)(J + K) = 0, \text{ only if } J_1 = 0 = K_1.$$

Multiplying out, we have in view of C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, that there exist a J and a K such that

$$J_1J + K_1K = 0, \text{ only if } J_1 = 0 = K_1.$$

By C_1, C_2 this may be written $J_3 + K_3 = 0$. Multiplying the last equation on the left by J' it appears that there exist a J and a K such that

$$J'J_3 + J'K_3 = 0, \text{ only if } J_1 = 0 = K_1.$$

Since $J'K_3 = 0$ (by C^j_{jk}, C^k_{jk}) we conclude there exists a J such that

$$J'J_3 = J'(J_1J) = 0, \text{ only if } J_1 = 0.$$

From C^j_l it follows that there exists a J such that

$$J_1J = 0, \text{ only if } J_1 = 0.$$

2. That C^k_l is dependent upon the remaining assumptions of R_{51} can be shown in a similar manner.†

3. That C^j_l is dependent upon the remaining assumptions of R_{59} is seen by interchanging k and j in 2 (i. e. in R_{51}).

4. That C^k_r is dependent upon the remaining assumptions of R_{68} is seen by interchanging j and k in 1 (i. e. in R_{50}).

---

* Also in R^4_{50}.

† The following proofs for 2, 3, 4 may also be employed. In R_{50} change *left* to *right*, obtaining thus R_{59}. In R_{59} interchange j and k, obtaining thus R_{61}.

5. In R_{71}, C^j_l is dependent upon the remaining assumptions. For, (Table I.) D_2, D_6 show that C_1 and C_2 are consequences of A_3, A_4, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_l (cf. R_1). C_l means that there exist a J and a K such that

$$(J+K)(J_1+K_1)=0, \text{ only if } J_1=0=K_1.$$

Multiplying out, we have in view of C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, that

$$JJ_1+KK_1=0, \text{ only if } J_1=0=K_1.$$

Multiplying on the left by J', in view of A_4, C^j_{jk}, C^k_{jk}, it appears that there exists a J such that

$$J'(JJ_1)=0, \text{ only if } J_1=0,$$

(or, only if $JJ_1=0$). This is C^j_l.

6. That C^j_r is dependent upon the remaining assumptions of R_{74} (cf. R_5) is seen by changing *left* to *right* in R_{71}.

7. That C^k_l is dependent upon the remaining assumptions of R_{75} is seen by interchanging j and k in R_{71} (cf. R_1).

8. That C^k_r is dependent upon the remaining assumptions of R_{78} is seen by interchanging j and k in R_{74} (cf. R_5).

§6.—*Semireducibility.*

An associative hypercomplex number system E, containing a modulus, is said to be *semireducible of the first kind* (Transactions, vol. 4 (October, 1903), pp. 437-444) when the following conditions are fulfilled:

$$A,\ C^j_{jk},\ C^j_{kj},\ C_1,\ C_2,\ C_r,\ C_l. \tag{7}$$

In this body of conditions, each one is independent of all the others.* In

*At the time this definition was framed I was not aware that a more general case had been studied by Molien in the Mathematische Annalen, vol. 41 (1893), pp. 92-93. When E satisfies the above conditions with C_1 omitted, it has according to Molien an *accompanying system.* Professor E. H. Moore has recently pointed out to me that when the conditions

$$A,\ C^j_{kj},\ C_2,\ C_r,\ C_l \tag{7a}$$

are satisfied, the group G of the system E will be reducible and will take the form

$$x'_{j_3}=\sum_{j_1}\Big\{\sum_{j_2}\gamma_{j_1j_2j_3}\,y_{j_2}+\sum_{k_2}\gamma_{j_1k_2j_3}\,y_{k_2}\Big\}x_{j_1},$$

$$x'_{k_3}=\sum_{j_1}\Big\{\sum_{j_2}\gamma_{j_1j_2k_3}\,y_{j_2}+\sum_{k_2}\gamma_{j_1k_2k_3}\,y_{k_2}\Big\}x_{j_1}+\sum_{k_1}\Big\{\sum_{j_2}\gamma_{k_1j_2k_3}\,y_{j_2}+\sum_{k_2}\gamma_{k_1k_2k_3}\,y_{k_2}\Big\}x_{k_3},$$

$$(j=1\ldots\ldots m;\ k=m+1\ldots\ldots n).$$

The above-mentioned definition of Molien is the special case obtained when $\gamma_{jkj}=0$ (C^j_{jk}). My definition of semireducibility of the first kind is the special case obtained from Molien's "nicht ursprüngliche" Systems when $\gamma_{jjk}=0$ (C_1). Since the conditions (7) are mutually independent, the Molien conditions, which are obtained by omitting C_1 from (7) must be independent and the conditions (7a) which are obtained from Molien's by omitting C^j_{jk} must evidently be independent also.

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order to make the independence proofs we add the following systems to those of §4.

X.

e_1	e_2	e_3
e_2	e_3	0
e_3	0	0

$E_j = e_1, e_2;\ E_k = e_3.$

XI.

e_1	e_2	e_3
e_2	e_1	e_3
e_3	$-e_3$	0

$E_j = e_1;\ E_k = e_2, e_3.$

A	C^j_{jk}	C^j_{kj}	C_1	C_2	C_r	C_l	Proof.
i	★	★	★	★	★	★	I.
★	i	★	★	★	★	★	VI.
★	★	i	★	★	★	★	VII. interchanging j and k.
★	★	★	i	★	★	★	X.
★	★	★	★	i	★	★	XI.
★	★	★	★	★	i	★	IX.
★	★	★	★	★	★	i	VIII.

§7.

It was shown in §1 that a hypercomplex number system is reducible when by a proper choice of the units it can be brought to the form

$$E \equiv E_j E_k = e_1 \ldots . e_m e_{m+1} \ldots . e_n,$$

where the following conditions are fulfilled:

A), associativity;
C_{jk}), $e_j e_k = 0$, i. e. every $\gamma_{jki} = 0$;
C_{kj}), $e_k e_j = 0$, i. e. every $\gamma_{kji} = 0$;
C_r), right-hand division possible; or
C_l), left-hand division possible.

In other words, the conditions

$$C_1), \qquad e_{j_1}e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3} \qquad (\gamma_{jjk} = 0),$$

$$C_2), \qquad e_{k_1}e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \qquad (\gamma_{kkj} = 0),$$

are both consequences of A, C_{jk}, C_{kj}, C_r (or C_l).

Suppose now that C_{jk}, C_{kj} are fulfilled, but C_l and C_r are not; in this case we have no reason to conclude that every γ_{jjk} and every γ_{kkj} is zero.* Nevertheless it can be demonstrated that there exist two hypercomplex number systems

$$F_j \equiv f_1 \;\ldots\; f_m, \; F_k = f_{m+1}, \ldots, f_n,$$

such that

$$f_{j_1} f_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} f_{j_3}, \; f_{k_1} f_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} f_{k_3}.$$

In particular, when either C_l on C_r is fulfilled, we obtain $f_i = e_i$ $(i = 1, \ldots, n)$.

In proof we have from the associativity condition

$$(e_{j_1} e_{j_2})\, e_{j_4} = e_{j_1} (e_{j_2} e_{j_4}),$$

$$\sum_{i_3=1}^{n} (\gamma_{j_1 i_2 i_3} \gamma_{i_3 j_4 i_5} - \gamma_{j_2 j_4 i_3} \gamma_{j_1 i_3 i_5}) = 0, \tag{8}$$

$$(j_1, j_2, j_4 = 1, \ldots, m; \; i_5 = 1, \ldots, n).$$

By C_{jk} and C_{kj} (7) becomes

$$\sum_{j_3=1}^{m} (\gamma_{j_1 j_2 i_3} \gamma_{j_3 j_4 i_5} - \gamma_{j_2 i_4 i_3} \gamma_{j_1 j_3 i_5}) = 0, \tag{9}$$

$$(j_1, j_2, j_4 = 1, \ldots, m; \; i_5 = 1, \ldots, n).$$

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* As an example of this possibility consider the system $\begin{array}{cc} e_1 + e_2 & 0 \\ 0 & 0. \end{array}$ It can easily be verified that the conditions A, C_{jk}, C_{kj} are fulfilled, but clearly $\gamma_{112} = 1$ $(\neq 0)$.

The m^3n equations (8) include the m^4 equations

$$\sum_{j_3=1}^{m} (\gamma_{j_1j_2j_3}\gamma_{j_3j_4j_5} - \gamma_{j_2j_4j_3}\gamma_{j_1j_3j_5}) = 0, \tag{10}$$

$$(j_1, j_2, j_4, j_5 = 1, \ldots, m),$$

and this shows that there exists a system in m units

$$F_j = f_1 \ldots f_m \text{ such that } f_{j_1}f_{j_2} = \sum_{j_3} \gamma_{j_1j_2j_3} f_{j_3}.$$

By interchanging j and k it follows that there exists a system in $m - n$ units

$$F_k = f_{m+1} \ldots f_n \text{ such that } f_{k_1}f_{k_2} = \sum_{k_3} \gamma_{k_1k_2k_3} f_{k_3}.$$

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